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Nonlinear Boundary Value Problems in Hilbert Spaces

J. W. LEE

Department of Mathematics, Oregon State University, Corvallis, Oregon 97331

AND

D. O'REGAN

Department of Mathematics and Applied Statistics, University of Idaho, Moscow, Idaho 83843

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In this paper we examine nonlinear two point boundary value problems, with Sturm–Liouville type boundary conditions, in a Hilbert Space. Our technique involves using the Topological Transversality Theorem of A. Granas which relies on the notions of an essential map and a priori bounds on solutions. We obtain existence results for a wide class of problems with our nonlinear term satisfying a Bernstein–Nagumo growth condition.

1. INTRODUCTION

In this paper, we establish the existence of a solution to the nonlinear two point boundary value problem

$$\begin{gathered} y''=f(t,y,y'), \qquad 0\leqslant t\leqslant 1, \\ y\in \mathbb{B}, \end{gathered} \tag{1.1}$$

where $\mathbb{B}$ denotes appropriate boundary conditions of Sturm–Liouville type. Here a solution y to (1.1) is a twice continuously differentiable function which takes its values in a real Hilbert Space $(H, \|\cdot\|)$. Thus, a solution $y\in C^2(I,H)$, where, in general, $C^k(I,H)$ is the space of functions $u\colon I=[0,1]\to H$ which have a continuous derivative of order k. $C(I,H)=C^0(I,H)$ is a Banach space with norm

$$\|u\|_0=\max\{\|u(t)\|: t\in I\}$$

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and $C^k(I, H)$ is a Banach space with norm

$$\|u\|_k = \max\{\|u\|_0, ..., \|u^{(k)}\|_0\}.$$

The nonlinear term $f: [0, 1] \times H \times H \to H$ is always assumed to be continuous. However, to establish that (1.1) has a solution, additional rather technical hypotheses must be placed on f (see, for example, [2, 3, 4, 9, 10]). In particular, [3] gives an existence theory based on monotonicity methods. Here the existence of a solution is obtained by a fixed point analysis which employs the Topological Transversality Theorem of Andrzej Granas; see [5, 6, 8] for a statement of this theorem and its application to scalar boundary value problems. The advantage of the present approach is that the existence of a solution to (1.1) can be established under more natural and less technical assumptions on the nonlinearity in (1.1).

We record for reference two standard results (see [2, 11]). As usual, we denote the inner product in H by $\langle \cdot, \cdot \rangle$; so $\|\cdot\|^2 = \langle \cdot, \cdot \rangle$.

PROPOSITION 1.1. *Suppose $u: I \to H$ is a differentiable function.*

(i) *If $u'(t) = 0$ for all $t \in I$, then u is a constant,*

(ii) *$d/dt\langle u(t), u(t)\rangle = 2\langle u'(t), u(t)\rangle$.*

PROPOSITION 1.2. *A subset M of $C(I, H)$ is relatively compact if and only if M is equicontinuous and the set $\{u(t): u \in M\}$ is relatively compact in H for each $t \in I$.*

We close this section by specifying the boundary conditions $\mathbb{B}$ which we allow in (1.1). $\mathbb{B}$ will denote either the boundary conditions (with $r, s \in H$)

$$-\alpha y(0) + \beta y'(0) = r, \qquad a y(1) + b y'(1) = s, \tag{1.2}$$

where α, β, a, $b \geqslant 0$, $\alpha + \beta > 0$, $a + b > 0$, and in addition

$$(\alpha + a)(\beta + b) > 0,$$

$$r = 0 \text{ if } \alpha = 0 \qquad \text{and} \qquad s = 0 \text{ if } \beta = 0,$$

or

$$y'(0) = r, \qquad y'(1) = s. \tag{1.3}$$

The added conditions in (1.2) exclude pure Dirichlet data at both ends, exclude pure Neumann data at both ends, and require that any pure Neumann condition is homogeneous. As a matter of notation, we denote by $C^2_{\mathbb{B}}(I, H)$ the subset of $C^2(I, H)$ which satisfies the boundary conditions $\mathbb{B}$. Thus, by definition a solution to (1.1) is an element of $C^2_{\mathbb{B}}(I, H)$.

2. General Existence Theorems

We consider the following restrictions on f: $[0, 1] \times H \times H \to H$:

$$f(t, u, p) \text{ is completely continuous;} \tag{2.1}$$

$$f(t, u, p) - u \text{ is completely continuous;} \tag{2.1'}$$

Given Ω a bounded subset of $C^2(I, H)$, there exist constants $\alpha > 0$, $A \geqslant 0$ such that (2.2)

$$\|f(t, u(t), u'(t)) - f(s, u(s), u'(s))\| \leqslant A\ |t-s|^\alpha$$

for all $u \in \Omega$ and $t, s \in I$.

The next two results are in the spirit of the continuity method of Leray and Schauder which reduces existence theorems for boundary value problems to the establishment of suitable a priori bounds.

THEOREM 2.1. *Let f: $[0, 1] \times H \times H \to H$ be continuous and satisfy* (2.1) *and* (2.2). *Let $\mathbb{B}$ denote the boundary conditions* (1.2). *Finally, assume there is a constant K such that $\|y\|_2 \leqslant K$ for any solution y to*

$$\begin{gathered} y'' = \lambda f(t, y, y'), \qquad 0 \leqslant t \leqslant 1, \\ y \in \mathbb{B}, \end{gathered} \tag{2.3$_\lambda$}$$

where $\lambda \in [0, 1]$. Then (1.1) *has at least one solution in $C^2(I, H)$.*

Proof. Define mappings

$$L, F\colon C^2_{\mathbb{B}}(I, H) \to C(I, H)$$

by

$$Ly = y'' \qquad \text{and} \qquad Fy(t) = f(t, y(t), y'(t)).$$

We observe first that F is completely continuous. Indeed, let Ω be a bounded set in $C^2_{\mathbb{B}}(I, H)$. Clearly (2.2) implies that $F(\Omega)$ is equicontinuous and $\{F\omega(t): \omega \in \Omega\} = \{f(t, \omega(t), \omega'(t)): \omega \in \Omega\}$ is relatively compact because f is completely continuous. Thus, $F(\Omega)$ is relatively compact by Proposition 1.2 and therefore F is completely continuous.

Next, we note that L is invertible. To this end, let L_0: $C^2_{\mathbb{B}_0}(I, H) \to C(I, H)$ be defined by $L_0 y = y''$ where $\mathbb{B}_0$ signifies the homogeneous (i.e., $r = s = 0$) boundary conditions corresponding to $\mathbb{B}$. It is elementary to check that L_0 is one-to-one and onto; hence, L_0^{-1} exists and is continuous either by the bounded inverse theorem or by explicit construction of the inverse as an integral operator. Likewise, the problem $Ly = 0$, $y \in \mathbb{B}$

can be solved explicitly for y; say, $y = l$. Thus, L^{-1} exists and is given by $L^{-1}g = L_0^{-1}g + l$.

Now set

$$U = \{u \in C^2_{\mathbb{B}}(I, H) \colon \|u\|_2 < K + \|l\|_2 + 1\},$$

which is an open set in the convex subset of $C^2_{\mathbb{B}}(I, H)$ of the Banach space $C^2(I, H)$. Define

$$H \colon \bar{U} \times [0, 1] \to C^2_{\mathbb{B}}(I, H)$$

by

$$\begin{aligned} H(u, \lambda) = H_\lambda(u) &= \lambda L^{-1}Fu + (1-\lambda)l \\ &= L_0^{-1}\lambda Fu + l. \end{aligned}$$

Evidently, H defines a compact homotopy. Also, the fixed points of H_λ are precisely the solutions of $(2.3)_\lambda$. Indeed, $H_\lambda u = u$ means $u \in \mathbb{B}$ and $L_0^{-1}\lambda Fu + l = u$; hence, $L^{-1}(\lambda Fu) = u$ and so $\lambda Fu = Lu$, i.e., $\lambda f(t, u, u') = u''$. Therefore, H_λ is fixed point free on ∂U by construction of U. Finally, $H_0(u) = l \in U$ is a constant map. Since H_0 is an essential map, the topological transversality theorem asserts that H_1 is essential. In particular, H_1 has a fixed point, i.e., $(2.3)_1$, or (1.1) has a solution. ∎

The operator L above is not invertible for the pure Neumann problem. To overcome this difficulty, we replace the differential equation in $(2.3)_\lambda$ by $y'' - y = \lambda[f(t, y, y') - y]$ and define L by $Ly = y'' - y$. Now, L is invertible when $\mathbb{B}$ specifies either (1.2) or (1.3) and the reasoning used to prove Theorem 2.1 leads to

THEOREM 2.2. *Suppose $f \colon [0, 1] \times H \times H \to H$ is continuous and satisfies* (2.1)′ *and* (2.2). *Assume there is a constant K such that $\|y\|_2 \leqslant K$ for each solution y to*

$$\begin{aligned} y'' - y = \lambda[f(t, y, y') - y], \qquad 0 \leqslant t \leqslant 1, \\ y \in \mathbb{B}, \end{aligned} \tag{2.4}_\lambda$$

where $\lambda \in [0, 1]$ and $\mathbb{B}$ denotes either the boundary condition (1.2) *or* (1.3). *Then* (1.1) *has at least one solution in $C^2(I, H)$.*

Remarks. (1) Theorems 2.1 and 2.2 hold with the same proofs if H is replaced by a Banach space.

(2) Theorems 2.1 and 2.2 give complimentary information even when $\mathbb{B}$ signifies the first set of boundary conditions (1.2), because (2.1) and (2.1)′ cannot hold simultaneously.

(3) Condition (2.1)′ can be replaced by $f(t, u, p) - cu$, completely continuous for some $c > 0$. Then the differential equation in $(2.4)_\lambda$ is replaced by

$$y'' - cy = \lambda[f(t, y, y') - cy].$$

In the following sections, we place further restrictions on f which ensure that the a priori bound K in Theorem 2.1 or 2.2 exists. Thus, we obtain existence of a solution to (1.1).

3. A Priori Bounds for $\|y\|_0$

Let f: $[0, 1] \times H \times H \to H$ be continuous and satisfy

$$\begin{gathered}\text{There is a constant } M > 0 \text{ such that} \\ \|y\| > M \text{ and } \langle y, p\rangle = 0 \text{ implies } \langle y, f(t, y, p)\rangle > 0.\end{gathered} \tag{3.1}$$

LEMMA 3.1. *Let f satisfy* (3.1) *and* $\mathbb{B}$ *denote the boundary data* (1.2). *Any solution $y = y(t)$ to* $(2.3)_\lambda$ *satisfies*

$$\|y\|_0 \leqslant M_0 = \max\left\{M, \frac{\|r\|}{\alpha}, \frac{\|s\|}{a}\right\},$$

where the term $\|r\|/\alpha$ or $\|s\|/a$ is omitted if α, respectively a, is zero.

Proof. Assume for the moment that $0 < \lambda \leqslant 1$. Let $y = y(t)$ be a solution of $(2.3)_\lambda$ and the set $\phi(t) = \frac{1}{2}\langle y(t), y(t)\rangle$. Suppose $\phi(t)$ achieves its maximum at t_0 in (0, 1). Then

$$\phi'(t_0) = \langle y(t_0), y'(t_0)\rangle = 0 \tag{3.2}$$

and

$$0 \geqslant \phi''(t_0) = \lambda\langle y(t_0), f(t_0, y(t_0), y'(t_0))\rangle + \|y'(t_0)\|^2. \tag{3.3}$$

Since $\lambda > 0$, these last two conditions and (3.1) imply $\|y(t_0)\| \leqslant M$. Thus, if $\|y(t)\|$ takes its maximum at a point t_0 in (0.1), then $\|y(t_0)\| \leqslant M$. Suppose $\|y(t)\|$ takes its maximum at $t_0 = 0$ and consider the boundary condition $-\alpha y(0) + \beta y'(0) = r$. If $\beta = 0$, then $\|y(0)\| = \|r\|/\alpha$. If $\beta > 0$ and $\alpha = 0$ then $r = 0$ and the boundary condition at 0 is $y'(0) = 0$. Consequently, $\phi'(0) = 0$ and $\phi''(0) > 0$ if $\|y(t_0)\| > M$. Then $\phi'(t) > 0$ for $t > 0$ and near zero, so $\phi(t)$ is increasing near 0, which contradicts the maximality of $\phi(0)$. We conclude

$\|y(0)\| \leqslant M$. It remains to consider the case $\beta > 0$ and $\alpha > 0$. Since $\phi(0)$ is the maximum of ϕ,

$$\begin{aligned} 0 \geqslant \beta\phi'(0) &= \langle y(0), \beta y'(0)\rangle = \langle y(0), \alpha y(0) + r\rangle \\ &= \alpha \|y(0)\|^2 + \langle y(0), r\rangle \\ &\geqslant \alpha \|y(0)\|^2 - \|y(0)\| \, \|r\| \\ &= \|y(0)\| \, [\alpha \|y(0)\| - \|r\|]. \end{aligned}$$

Consequently, $\|y(0)\| \leqslant \|r\|/\alpha$ and if $\|y(t)\|$ takes its maximum at 0, then $\|y(0)\| \leqslant \max\{M, \|r\|/\alpha\}$. Likewise, if $\|y(t)\|$ takes its maximum at 1, then $\|y(1)\| \leqslant \max\{M, \|s\|/a\}$. Thus, Lemma 3.1 is confirmed if $0 < \lambda \leqslant 1$.

Finally, suppose $\lambda = 0$ in $(2.3)_\lambda$. Then $y(t) = tA + B$ for some $A, B \in H$, If $\|y(t)\|$ takes its maximum at t_0 in $(0, 1)$, then (3.3) with $\lambda = 0$ yields $y'(t_0) = 0$. Then $y(t) \equiv B$ and the boundary conditions give $-\alpha B = r$ and $aB = s$. Since $\alpha + a > 0$, we have either $B = -r/\alpha$ or $B = s/a$ and $\|y(t)\| \leqslant M_0$ in this case. Next, suppose the maximum of $\|y(t)\|$ occurs at $t_0 = 0$. As before, if $\beta = 0$, then $\|y(0)\| = \|r\|/\alpha$, while if $\beta > 0$ and $\alpha = 0$ then $r = 0$ and $y'(0) = 0$. Then $y(t) \equiv B$ and the boundary condition at $t = 1$ gives $aB = s$. Since $\alpha = 0$, we must have $a > 0$ and so $\|y(t)\| = \|s\|/a$. It remains to treat the case $\beta > 0$ and $\alpha > 0$. Since $\phi(0)$ is a maximum the previous argument for this case applies as is and yields $\|y(0)\| \leqslant \|r\|/\alpha$. Combining cases, we find that when the maximum occurs at $t_0 = 0$, then $\|y(t_0)\| \leqslant \max\{\|s\|/a, \|r\|/\alpha\}$. By symmetry, the same bound holds if $\|y(t)\|$ takes its maximum at $t_0 = 1$. This completes the proof of Lemma 3.1. ∎

Now, consider the family of problems $(2.4)_\lambda$. The differential equation is $y'' = \lambda f(t, y, y') + (1-\lambda) y$. Consequently, if $\phi(t) = \frac{1}{2}\langle y, y\rangle$ as above, we find

$$\phi(t) = \langle y, y'\rangle$$

and

$$\phi''(t) = \lambda\langle y, f\rangle + (1-\lambda)\langle y, y\rangle + \langle y', y'\rangle.$$

Reasoning in essentially the same way as for Lemma 3.1, we obtain

LEMMA 3.2. *Let f satisfy* (3.1) *and $\mathbb{B}$ denote the boundary date* (1.2). *Then any solution $y = y(t)$ to $(2.4)_\lambda$ satisfies the bound in Lemma* 3.1.

Consider the Neumann data (1.3). If $r = s = 0$ then the arguments pertaining to $y'(0) = 0$ in Lemma 3.1 show that $\|y(0)\| \leqslant M$ if $\|y(t)\|$ takes its maximum at $t_0 = 0$. The same bound holds at $t_0 = 1$ when the homogeneous Neumann condition $y'(1) = 0$ is given. The reasoning of

Lemma 3.1 also shows that $\|y(t_0)\| \leqslant M$ if $\|y(t)\|$ is maximized at t_0 in $(0, 1)$. Thus, we have

LEMMA 3.3. *Let f satisfy* (3.1). *Then any solution $y = y(t)$ to* $(2.3)_\lambda$ *or to* $(2.4)_\lambda$ *with $\mathbb{B}$ denoting homogeneous Neumann data $y'(0) = y'(1) = 0$ satisfies* $\|y\|_0 \leqslant M$.

We defer the discussion of inhomogeneous Neumann data to Section 6. Already in the case of scalar differential equations the inhomogeneous Neumann problem behaves in an essentially different manner than the corresponding homogeneous problem (see [6]).

Remark. The arguments used to prove Lemmas 3.1 and 3.2 hold if $\mathbb{B}$ denotes the pure Dirichlet data, $y(0) = r$, $y(1) = s$. Thus, the a priori bound for Lemma 3.1 holds for Dirichlet data.

4. A PRIORI BOUNDS FOR $\|y'\|_0$ AND $\|y''\|_0$

In this section we establish an a priori bound for $\|y'\|_0$ given an a priori bound for $\|y\|_0$ and some control on the growth of the nonlinearity $f(t, u, p)$ in p. Specifically, we assume f satisfies the Bernstein–Nagumo type condition:

There is an increasing function $\psi\colon [0, \infty) \to (0, \infty)$ such that $1/\psi$ is integrable on $[0, \infty)$,

$$\|f(t, u, p)\| \leqslant \psi(\|p\|)$$

for $(t, \|u\|)$ in $[0, 1] \times [0, M_0]$ and

$$\int_c^\infty \frac{dx}{\psi(x)} > 1, \tag{4.1}$$

where

$$c = \min\left\{\frac{\|r\| + \alpha M_0}{\beta}, \frac{\|s\| + aM_0}{b}\right\}.$$

LEMMA 4.1. *Consider the family of problems* $(2.3)_\lambda$ *where $\mathbb{B}$ denotes either the boundary conditions* (1.2) *or* (1.3). *Assume there is a constant M_0 so that $\|y\|_0 \leqslant M_0$ for each solution y to* $(2.3)_\lambda$ *and that f satisfies* (4.1). *Then there is a constant M_1 independent of λ such that $\|y'\|_0 \leqslant M_1$ for all solutions y to* $(2.3)_\lambda$.

Proof. Let y be a solution to $(2.3)_\lambda$. Either β or b is positive. If $\beta > 0$ we find

$$\|y'(0)\| \leqslant \frac{\|r\| + \alpha M_0}{\beta},$$

while if $b > 0$,

$$\|y'(1)\| \leqslant \frac{\|s\| + aM_0}{b}.$$

By continuity, there is a point τ in $[0, 1]$ such that

$$\|y'(\tau)\| \leqslant c = \min\left\{\frac{\|r\| + \alpha M_0}{\beta}, \frac{\|s\| + aM_0}{b}\right\}.$$

From $(2.3)_\lambda$ and (4.1) we have

$$\|y''\| = \|\lambda f(t, y, y')\| \leqslant \psi(\|y'\|).$$

If $\|y'(t)\| \neq 0$, Proposition 1.1 and the Schwarz inequality give

$$\|y'\|' = \frac{\langle y', y''\rangle}{\|y'\|} \leqslant \|y''\|.$$

Combine this estimate with the previous inequality to obtain

$$\|y'\|' \leqslant \psi(\|y'\|) \tag{4.2}$$

at any point t where $\|y'(t)\| \neq 0$. Now, suppose $\|y'(t)\| > c$ for some t in $[0, 1]$. Since $\|y'(\tau)\| \leqslant c$ and y' is continuous, there is an interval $d \leqslant s \leqslant t$ (or $t \leqslant s \leqslant d$) such that $\|y'(s)\| > 0$ and $\|y'(d)\| = c$. To be definite, suppose the interval is $d \leqslant s \leqslant t$. Then by (4.2)

$$\int_d^t \frac{\|y'(s)\|'}{\psi(\|y'(s)\|)}\, ds \leqslant t - d \leqslant 1,$$

and

$$\int_c^{\|y'(t)\|} \frac{du}{\psi(u)} \leqslant 1 < \int_c^\infty \frac{du}{\psi(u)},$$

by (4.1). This last inequality clearly implies the existence of $M_1' < \infty$ (and independent of λ) such that $\|y'(t)\| \leqslant M_1'$. We conclude

$$\|y'\|_0 \leqslant \max\{c, M_1'\} \equiv M_1. \quad \blacksquare$$

Remark. In the case of linear homogeneous Neumann data $\alpha = r = a = s = 0$, so $c = 0$ in (4.1).

Entirely similar reasoning can be applied to the family of problems $(2.4)_\lambda$. In this case, if M_0 is an a priori bound on solutions y of $(2.4)_\lambda$, so that $\|y\|_0 \leqslant M_0$, then

$$\begin{aligned} \|y''\| &\leqslant \lambda \, \|f(t, y, y')\| + (1-\lambda) \, \|y\| \\ &\leqslant \max\{\psi(\|y'\|), M_0\} \equiv \psi_0(\|y'\|) \end{aligned}$$

if f satisfies (4.1). Now, the arguments used to prove Lemma 4.1 yield

LEMMA 4.2. *Consider the family of problems* $(2.4)_\lambda$ *where* $\mathbb{B}$ *denotes either the boundary conditions* (1.2) *or* (1.3). *Assume there is a constant* M_0 *such that* $\|y\|_0 \leqslant M_0$ *for each solution* y *to* $(2.4)_\lambda$ *and that* f *satisfies* (4.1) *with* ψ *replaced by* ψ_0 *in the integral condition. Then there is a constant* M_1 *(independent of* λ*) so that* $\|y'\|_0 \leqslant M_1$ *for all solutions* y *to* $(2.4)_\lambda$.

Of course, under the hypotheses in Lemmas 4.1 and 4.2 we automatically obtain an a priori bound for y'', namely,

$$\|y''\|_0 \leqslant \max_{0 \leqslant p \leqslant M_1} \psi(p) \qquad \text{or} \qquad \|y''\|_0 \leqslant \max_{0 \leqslant p \leqslant M_1} \psi_0(p)$$

for problems $(2.3)_\lambda$ and $(2.4)_\lambda$, respectively.

5. EXISTENCE OF SOLUTIONS FOR STURM–LIOUVILLE DATA, EXCLUDING PURE DIRICHLET AND PURE NEUMANN CONDITIONS

Combining the results in Sections 2, 3, and 4 we obtain:

THEOREM 5.1. *Consider the boundary value problem* (1.1) *where* $\mathbb{B}$ *denotes the boundary conditions* (1.2). *Assume the nonlinearity* $f\colon [0, 1] \times H \times H \to H$ *satisfies* (2.1), (2.2), (3.1), *and* (4.1). *Then* (1.1) *has at least one solution.*

THEOREM 5.2. *Consider* (1.1) *with the boundary condition* $\mathbb{B}$ *given by* (1.2). *Assume* f *satisfies* (2.1)$'$, (2.2), (3.1), *and* (4.1) *with* ψ *replaced by* ψ_0 *in the integral condition. Then* (1.1) *has at least one solution.*

6. COMMENTS ON THE PURE NEUMANN AND DIRICHLET PROBLEMS

In the case of homogeneous Neumann data $y'(0) = 0$, $y'(1) = 0$, the results of Sections 2, 3, and 4 immediately yield the following counterpart to Theorem 5.2.

THEOREM 6.1. *Consider* (1.1) *with homogeneous Neumann boundary conditions. Assume f satisfies* (2.1)′, (2.2), (3.1), *and* (4.1) *with ψ replaced by ψ_0 in the integral condition. Then* (1.1) *has at least one solution.*

As noted earlier (see [6]), the analogue of Theorem 6.1 for a scalar differential equation with inhomogeneous Neumann data does not hold. Further restrictions on f are needed to guarantee existence of a solution to (1.1) when $\mathbb{B}$ specifies (1.3) and r and/or s is nonzero. We consider the following additional restrictions on f:

f is coercive in u; that is, there is a constant $\gamma > 0$ such that (6.1)

$$\langle u-v, f(t,u,p)-f(t,v,p)\rangle \geqslant \gamma \, \|u-v\|^2$$

and

$f(t,u,p)$ is bounded in p when (t,u) vary in a bounded set. (6.2)

THEOREM 6.2. *Consider* (1.1) *with inhomogeneous Neumann boundary conditions. Assume f satisfies* (2.1)′, (2.2), (6.1), *and* (6.2). *Then* (1.1) *has at least one solution.*

Proof. Define $\sigma = \sigma(t) = (t^2/2)(s-r) + tr$ so that

$$\sigma'(0) = r \qquad \text{and} \qquad \sigma'(1) = s.$$

Set $u = y - \sigma$. Then y satisfies (1.1) with boundary data $y'(0) = r$, $y'(1) = s$ if and only if u satisfies

$$\begin{gathered} u'' = f(t, u+\sigma, u'+\sigma') + r - s \equiv F(t,u,u') \\ u'(0) = 0, \; u'(1) = 0. \end{gathered} \tag{6.3}$$

We confirm that F satisfies the hypotheses of Theorem 6.1. Clearly (2.1)′ and (2.2) hold for F. To confirm (3.1) for F use (6.2) to secure a constant C such that

$$\|f(t, \sigma(t), p)\| \leqslant C$$

for all p. Then from (6.1)

$$\begin{aligned} \langle u, F(t,u,p)\rangle &= \langle u, f(t, u+\sigma, u'+\sigma') - f(t, \sigma, u'+\sigma')\rangle \\ &\quad + \langle u, f(t, \sigma, u'+\sigma')\rangle + \langle u, r-s\rangle \\ &\geqslant \gamma \, \|u\|^2 - (C + \|r-s\|) \, \|u\| > 0, \end{aligned}$$

provided $\|u\| > \{C + \|r-s\|\}/\gamma$. Thus, (3.1) holds with $M = \{C + \|r-s\|\}/\gamma$.

Next, (4.1) holds with $M_0 = M$ and $\psi \equiv B > 0$ a constant, in view of (6.2). Then $\psi_0 = B + M$ and the integral in (4.1) diverges. Thus (4.1) holds, and (6.3) has a solution u. Then $y = u + \sigma$ solves (1.1) with the inhomogeneous Neumann data $y'(0) = r$, $y'(1) = s$. ∎

The methods in this paper do not allow us to establish existence for the pure Dirichlet boundary value problem. The problem is that the boundary conditions $y(0) = r$, $y(1) = s$ do not imply (except in one dimension) the existence of a constant $K > 0$ and a τ in $[0, 1]$, with τ dependent on y, such that $\|y'(\tau)\| \leqslant K$. For instance, in two dimensions the functions $y = (\cos 2\pi nt, \sin 2\pi nt)$ satisfy $y(0) = y(1) = (1, 0)$ but $\|y'(t)\| = 2\pi n$ and no bound K as above exists. Thus, the reasoning used in Section 4 to obtain an a priori bound on $\|y'\|_0$ breaks down.

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